Single and two-particle motion of heavy particles in turbulence

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We study motion of small particles in turbulence when the particle relaxation time falls in the range of inertial time-scales of the flow. Due to inertia, particles drift relative to the fluid. We show that the drift velocity is close to the Lagrangian velocity increments of turbulence at the particle relaxation time. We demonstrate that the collective drift of two close particles makes them see local velocity increments fluctuate fast and we introduce the corresponding Langevin description for separation dynamics. This allows to describe the behavior of the Lyapunov exponent and give the analogue of Richardson's law for separation above viscous scale.

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Motion of small particles in a fluid, due to random molecular forces, is the subject of the classical theory of Brownian motion. Scale separation between the particle relaxation time and time-scales of the forces allows to introduce an effective Langevin description of the driving force as white noise in time [1]. In contrast, here we consider the situation where the random driving force originates not from the microscopic motions, but rather from the macroscopic turbulent motion of the surrounding fluid [2]. The limit of the particle relaxation time much larger than the characteristic time-scales of turbulence (very heavy particles) can be described as in the Brownian motion case [3]. In the opposite limit, when particle relaxation time is much smaller than the characteristic time-scales of turbulence (very large friction), particles follow the flow closely and the two-particle dispersion – of interest to us here, – is approximately the same as for fluid particles. In this Letter we study the intermediate case of heavy particles, where the relaxation time falls in the range of flow time-scales corresponding to the inertial interval of turbulence. This precludes Langevin description for the single-particle motion. However, for *two* particles, because of their collective drift relative to the fluid, velocity increments determining the separation do vary fast. This allows to introduce effective Langevin description for the dynamics of separations. The description enables us to find several results on particle behavior in turbulence valid beyond Kolmogorov theory.

Behavior of small inertial particles in turbulence has received much attention lately [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. This problem has many applications including rain formation in clouds [4, 5], ocean physics [6] and engineering [7]. Theoretical study of the problem mostly involved modeling turbulence by a white noise in time, Gaussian velocity field, known as Kraichnan model. Even in that case theoretical study is rather difficult, analytic results were mainly obtained for the one-dimensional case [8, 9]. The limit of heavy particles, considered here for turbulence, was studied numerically for Kraichnan model in [3, 10]. For turbulence, numerical studies of intermediate regime of moderately heavy particles were performed in [11, 12, 13].

We consider the motion of a small spherical particle in an incompressible, statistically steady, turbulent flow $\boldsymbol{u}(\boldsymbol{x},t)$. We assume that the drag force acting on the particle obeys Stokes' law. Designating the particle position and velocity by $\boldsymbol{x}(t)$ and $\boldsymbol{v}(t)$, Newton's law reads

$$\dot{\boldsymbol{x}} = \boldsymbol{v}\ , \qquad \dot{\boldsymbol{v}} = -\frac{\boldsymbol{v}-\boldsymbol{u}(\boldsymbol{x}(t),t)}{\tau}\,. \tag{1}$$

Here $\tau=(2/9)(\rho_0/\rho)(a^2/\nu)$, where ρ_0 and a are the particle density and radius, while ρ and ν are the fluid's density and kinematic viscosity. We briefly review relevant properties of $\boldsymbol{u}(\boldsymbol{x},t)$, see e.g. [2] for details. Velocity field, excited at the integral scale L, fluctuates in a wide (inertial) range of spatial scales $\eta \ll l \ll L$. The characteristic velocity u_l of fluctuations at a scale l is related to the temporal scale t_l by $u_l t_l/l \sim 1$. At the viscous scale η we have $t_\eta \sim \eta^2/\nu$. For moderate Reynolds numbers one can use Kolmogorov theory (below K41) that gives $u_l \sim (\epsilon l)^{1/3}$, $t_l \sim \epsilon^{-1/3} l^{2/3}$ and $\eta \sim (\nu^3/\epsilon)^{1/4}$, where ϵ is the mean energy injection rate. Eq. (1) is valid if $\eta \gg a$ and inertia-induced particle drift relative to the flow, described by $\boldsymbol{w}(t) \equiv \boldsymbol{v}(t)-\boldsymbol{u}(\boldsymbol{x}(t),t)$, has small Reynolds number wa/ν [16]. We shall consider $t_\eta \ll \tau \ll t_L$, which implies that particles are heavy, $\rho_0/\rho \sim (\eta^2/a^2)(\tau/t_\eta) \gg 1$, justifying the neglect of such effects as added mass in Eq. (1) [16]. Beyond K41, quantities like η and w have strong spatiotemporal fluctuations, and in that setup we will refer to their local (in space and time) values on statistically relevant events.

We first consider the drift velocity $\boldsymbol{w}$. From Eq. (1), in the steady state, $\boldsymbol{w}=\int_{-\infty}^{0}\delta_t^P \boldsymbol{u}\exp(t/\tau)dt/\tau$, where $\delta_t^P\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{x}(t),t)-\boldsymbol{u}(\boldsymbol{0},0)$ is the turbulent velocity difference in particle frame and we chose $t=0$ and $\boldsymbol{x}(0)=0$. The difference $\delta_t^P\boldsymbol{u}$ is analogous to the Lagrangian difference $\delta_t^L\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{q}(t),t)-\boldsymbol{u}(\boldsymbol{0},0)$ where $\boldsymbol{q}(t)$ is a fluid particle trajectory obeying $\dot{\boldsymbol{q}}=\boldsymbol{u}(\boldsymbol{q}(t),t)$ and $\boldsymbol{q}(0)=\boldsymbol{0}$. Just as $\delta_t^L u$, the increment $\delta_t^P u$ should be, on a rough scale, a non-decreasing function of $|t|$, growing at most as a power law. Then the integral for $\boldsymbol{w}$ yields the order of magnitude estimate $w \sim \delta_\tau^P u$ and we also see that the charac-

teristic time of variations of $\boldsymbol{w}$ is τ. Let us show that from $w \sim \delta_\tau^P u$ one can pass to $w \sim \delta_\tau^L u$, that is $\delta_\tau^P u \sim \delta_\tau^L u$. We write the telescopic sum $\delta_t^P \boldsymbol{u} = \delta_t^L \boldsymbol{u} + \delta \boldsymbol{v}$, where $\delta \boldsymbol{v} \equiv \boldsymbol{u}(\boldsymbol{x}(t),t) - \boldsymbol{u}(\boldsymbol{q}(t),t)$ is the velocity difference of the separating fluid and inertial particles. The separation is due to the combined effect of the inertial drift and the explosive separation of trajectories in the inertial range [19], so that $\delta v \lesssim max[w, \delta_t^L u]$, where $\delta_t^L u$ is the relative velocity of explosively separating particles [2]. Thus, at t such that $w \lesssim \delta_t^L u$, one has $\delta v \lesssim \delta_t^L u$ and, then, from the telescopic sum, $\delta_t^P u \sim \delta_t^L u$. This, combined with $w \sim \delta_\tau^P u$, allows to show $w \sim \delta_\tau^L u$. Since $\delta_t^L u$ roughly grows with t at $0 < t < t_L$ and $0 = \delta_{t=0}^L u < w \lesssim \delta_{t_L}^L u$ (the latter is just an upper bound for w), there exists t_* such that $w \sim \delta_{t_*}^L u$. At this t_*, from the previous considerations one has $\delta_{t^*}^P u \sim \delta_{t^*}^L u \sim w \sim \delta_\tau^P u$. Finally $\delta_t^P u$ roughly increases monotonically for timescales less than $L/w \gtrsim L/\delta_{t_L}^L u \sim t_L$, so we conclude that $t^* \sim \tau$, whence $w \sim \delta_\tau^L u$. This means that particles follow only the flow fluctuations with time-scales larger than τ. Though expectable, the result uses specific properties of turbulence and does not hold for any $\boldsymbol{u}(\boldsymbol{x},t)$, where the relation between the spatial and temporal fluctuations is different. For the particle velocity, from $w \ll u$, we have $\boldsymbol{v} \approx \boldsymbol{u}(\boldsymbol{x}(t),t)$. Note that $\boldsymbol{w}$ is accessible experimentally through acceleration $\boldsymbol{a} \equiv \dot{\boldsymbol{v}} = -\boldsymbol{w}/\tau$. Also note that $w \sim \delta_\tau^L u$ applies at any τ: at $\tau \gtrsim t_L$ it is trivial, while at $\tau \ll t_\eta$ one has $\boldsymbol{w} \approx -\tau[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}] \approx \delta_\tau^L \boldsymbol{u}$ [15].

The local equality $w \sim \delta_\tau^L u$ suggests that the time-averages along the particle trajectory $\boldsymbol{x}(t)$ satisfy

$$\tau^n \langle a^n \rangle = \langle w^n \rangle \sim \langle (\delta_\tau^L u)^n \rangle \sim u_L^n (\tau/t_L)^{\gamma_n} . \qquad (2)$$

The anomalous exponents γ_n could differ from their counterparts for the more usual $\langle(\delta_\tau^L u)^n\rangle_q$, where the averaging is along the fluid particle trajectory [2, 20], – inertial particles tend to concentrate preferentially in specific regions of the flow [4]. Yet the difference seems unimportant: introducing l_τ by $t_{l_\tau} \sim \tau$, one has $\langle(\delta_\tau^L u)^n\rangle \sim \langle[\delta u(l_\tau)]^n\rangle$, where $\delta\boldsymbol{u}(\boldsymbol{R}) \equiv \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{R}) - \boldsymbol{u}(\boldsymbol{x})$, and $\langle[\delta u(l_\tau)]^n\rangle$ allows additional spatial averaging over l_τ vicinity of $\boldsymbol{x}(t)$, beyond which the preferential concentration is expected to be small [13]. In K41, similarity of properties of w and $\delta_\tau^L u$ follows by dimensional analysis: one has $w \sim \sqrt{\epsilon\tau}$ and $\langle w^n\rangle \sim (\epsilon\tau)^{n/2}$ like for $\delta_\tau^L u$ [20].

We now consider two-particle motion. We assume the particle separation $\boldsymbol{R} = \boldsymbol{x}' - \boldsymbol{x}$ much larger than radius a, so that hydrodynamic interactions between particles are negligible. Then each particle satisfies Eq. (1), producing

$$\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}/\tau = \delta\boldsymbol{u}(\boldsymbol{R})/\tau . \qquad (3)$$

At $R \gg l_\tau$ particle dispersion, driven by $\delta\boldsymbol{u}(\boldsymbol{R}) \sim u_R$, is determined by turbulent fluctuations slower than τ, and the separation is like for fluid particles: $\dot{\boldsymbol{R}} = \boldsymbol{v}' - \boldsymbol{v} = \delta\boldsymbol{u}(\boldsymbol{R}) + \boldsymbol{w}' - \boldsymbol{w} \approx \delta\boldsymbol{u}(\boldsymbol{R})$, by $w \sim u_{l_\tau} \ll u_R$. In contrast, at $R \ll l_\tau$ dispersion laws peculiar for inertial particles hold (in K41 $l_\tau \sim \epsilon^{1/2}\tau^{3/2} \sim \eta(\tau/t_\eta)^{3/2}$). We first treat $R(t) \ll \eta$ where $\delta u_i(\boldsymbol{R}) \approx R_j \nabla_j u_i(\boldsymbol{x}(t),t)$ and

$$\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}/\tau = (\boldsymbol{R}\cdot\boldsymbol{\nabla})\boldsymbol{u}/\tau . \qquad (4)$$

The main characteristics of Eq. (4) is the Lyapunov exponent $\lambda_1 = \lim_{t\to\infty} \ln[p(t)/p(0)]/t = \langle \dot{p}/p\rangle$ describing the exponential growth of the distance $\boldsymbol{p} \equiv (\boldsymbol{R}, \tau\dot{\boldsymbol{R}})$ between two infinitesimally close trajectories in the phase space of Eq. (1). Here λ_1 also describes the growth of $\boldsymbol{R}$ and $\dot{\boldsymbol{R}}$ separately, so it can be observed via spatial trajectories of close particles with small relative velocity.

We assume that the time-average $\langle \dot{p}/p\rangle$ can be found by averaging over the statistics of turbulence: $\lambda_1 = \lim_{t\to\infty}\langle \dot{p}(t)/p(t)\rangle_u$. It is useful to consider first the Kraichnan model where $\nabla_j u_i$ in Eq. (4) is modeled by white noise $\hat\sigma_{ij}$ obeying $\langle\hat\sigma_{ij}(t)\hat\sigma_{mn}(t')\rangle = D\delta(t-t')\,[(d+1)\delta_{im}\delta_{jn} - \delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}]$ where d is the space dimension [6]. Passing to dimensionless time $s \equiv Dt$, one finds $(D\tau)\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}} = \sigma'(s)\boldsymbol{R}$, where $\langle\sigma'_{ij}(s_1)\sigma'_{mn}(s_2)\rangle = \delta(s_1-s_2)\,[(d+1)\delta_{im}\delta_{jn} - \delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}]$. At $D\tau \ll 1$ one may drop $(D\tau)\ddot{\boldsymbol{R}}$. The resulting equation is the same as for separation of fluid particles so $\lambda_1 \approx \lambda_1^{fl}$, where $\lambda_1^{fl} \sim D$ is the Lyapunov exponent of fluid particles [19]. On the other hand, using dimensionless time $s' = D^{1/3}t/\tau^{2/3}$, one finds $\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}/(D\tau)^{1/3} = \sigma'(s')\boldsymbol{R}$, cf. [10]. At $(D\tau)^{1/3} \gg 1$ one may drop $\dot{\boldsymbol{R}}/(D\tau)^{1/3}$. Then, by dimensional analysis, $\lambda_1 \sim D^{1/3}/\tau^{2/3}$. We observe that at $\lambda_1\tau \ll 1$ one can drop the first, inertial, term in $\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}/\tau = \hat\sigma\boldsymbol{R}/\tau$, while at $\lambda_1\tau \gg 1$, the friction term can be dropped: the characteristic time of variations of $\boldsymbol{R}$ is λ_1^{-1} so the ratio of $\ddot{\boldsymbol{R}}$ to $\dot{\boldsymbol{R}}/\tau$ is estimated as $\lambda_1\tau$. The region of small inertia $\lambda_1\tau \sim D\tau \ll 1$ is separated from the region of large inertia $\lambda_1\tau \sim (D\tau)^{1/3} \gg 1$ by the long crossover region $D\tau \gtrsim (D\tau)^{1/3} \sim 1$ where $\ddot{\boldsymbol{R}} \sim \dot{\boldsymbol{R}}/\tau$ and $\lambda_1\tau \sim 1$. This explains the numerical results of [3, 10]. For example, in $d=2$ one may write $\lambda_1\tau = [D\tau]^{1/3}\tilde\lambda_1[(D\tau)^{-1/3}]$, where $\tilde\lambda_1(\epsilon) = \langle\Re z\rangle$ for complex dynamics $\dot z = -z^2 - \epsilon z + \gamma$ [6]. Here uncorrelated noises γ_i obey $\langle\gamma_1(t)\gamma_1(t')\rangle = \delta(t-t')$ and $\langle\gamma_2(t)\gamma_2(t')\rangle = 3\delta(t-t')$. The crossover region is described by slow, order unity, variation of $\tilde\lambda_1(\epsilon)$ from $\tilde\lambda(1) \approx 0.5$ to $\lambda^0 \equiv \tilde\lambda_1(0) \approx 2$ [3], and $\lambda_1 \approx \lambda^0 D^{1/3}/\tau^{2/3}$ at $(D\tau)^{1/3} \gg 1$. The time-scale beyond which $\langle\dot p(t)/p(t)\rangle$ relaxes to its steady-state value λ_1, forgetting the initial conditions (for $d=2$, the relaxation time of z), can be estimated as λ_1^{-1}. Indeed, λ_1^{-1} is the only time-scale both at $D\tau \ll 1$ and $(D\tau)^{1/3} \gg 1$, while at $(D\tau)^{1/3} \sim 1$ all coefficients in $\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}/(D\tau)^{1/3} = \sigma'(s')\boldsymbol{R}$ are of order unity so again $\tau^{2/3}/D^{1/3} \sim \lambda_1^{-1}$ is the only possible time-scale.

Let us now consider the turbulent velocity gradient *seen by the particle*, $\nabla_j u_i(\boldsymbol{x}(t),t)$ in Eq. (4). It is determined by fluctuations at the viscous scale, $\nabla u \sim u_\eta/\eta$, while its time variation can be inferred from

$$\frac{d}{dt}\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{x}(t),t) = [(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}) + \boldsymbol{w}\cdot\boldsymbol{\nabla}]\,\boldsymbol{\nabla}\boldsymbol{u}. \qquad (5)$$

At $\tau \gg t_\eta$, the drift derivative $(\boldsymbol{w}\cdot\boldsymbol{\nabla})\nabla u \sim \nabla u(w/\eta)$ dominates the substantial derivative $(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{\nabla u} \sim \nabla u(u_\eta/\eta)$. Thus $\nabla_j u_i(\boldsymbol{x}(t),t)$ varies at time-scale η/w – during this time particle deviates from the carrying flow by the spatial scale of variations of velocity gradient, η.

Below we study λ_1 as a function of the Stokes number $\mathrm{St} \equiv \lambda_1^{turb}\tau$. Here λ_1^{turb} is the Lyapunov exponent of fluid particles in turbulence. K41 dimensional analysis gives $\lambda_1^{turb} \sim \sqrt{\epsilon/\nu}$, while it does not fix λ_1 due to the additional time-scale τ. At physically relevant Reynolds numbers $\lambda_1^{turb} \sim \sqrt{\epsilon/\nu}$ is valid [12], indicating that λ_1^{turb} is determined by weakly intermittent events, and implying the same for λ_1 at least for small St where $\lambda_1 \approx \lambda_1^{turb}$. We shall assume that λ_1 is determined by weakly intermittent events for any St and use K41 for order of magnitude estimates, justifying this later (e. g. t_η and η will refer to their K41 values). The domain of applicability of $\lambda_1 \approx \lambda_1^{turb}$ is $\mathrm{St} \ll 1$ – at these St the drift contribution in Eq. (5) is negligible, while $\lambda_1\tau \ll 1$ allows to neglect the inertial term in Eq. (4). We now consider $\mathrm{St} \gg 1$. The idea of the analysis is to make the natural assumption $\lambda_1 \ll \lambda_1^{turb}$, easily verifiable in the end of the calculation, and to use that the timescale λ_1^{-1} of variations of $\boldsymbol{R}$ is much larger than the time-scale of variations of $\boldsymbol{\nabla u}$ in Eq. (4): $\lambda_1^{-1} \gg [\lambda_1^{turb}]^{-1} \gtrsim \eta/w$. Then the effect of $\nabla\boldsymbol{u}$ on $\boldsymbol{R}$ can be represented by a white noise.

The first step is the derivation of the Lyapunov exponent $\lambda_1(a)$ for the auxiliary constant-drift problem $\ddot{R}_i + \dot{R}_i/\tau = R_j\nabla_j u_i(\boldsymbol{x_a}(t),t)/\tau$, where $\dot{\boldsymbol{x}}_{\boldsymbol{a}} = \boldsymbol{u}(\boldsymbol{x_a}(t),t) + \boldsymbol{a}$. Having in mind the application, the constant vector $\boldsymbol{a}$ is assumed to have the same characteristic value $\sqrt{\epsilon\tau}$ as w, so that the correlation time τ_c of $\boldsymbol{\nabla u}(\boldsymbol{x_a}(t),t)$ is $\eta/a \sim 1/\lambda_1^{turb}\sqrt{\mathrm{St}}$. Averaging Eq. (4) over time Δt satisfying $\tau_c \ll \Delta t \ll \lambda_1^{-1}$, we find $\ddot{R}_i + \dot{R}_i/\tau = R_j\overline{\nabla}_j u_i/\tau$, where $\overline{\nabla}_j u_i \equiv \int_t^{t+\Delta t}\nabla_j u_i(\boldsymbol{x_a}(t'),t')\,dt'/\Delta t$. We observe that $\overline{\nabla}_j u_i$ is a Gaussian process with zero mean and pair correlation which – due to stationarity, spatial homogeneity of small-scale turbulence and incompressibility, – is determined by $F_{ijmn}(\boldsymbol{a}) \equiv \int dt\,\langle\nabla_j u_i(\boldsymbol{0},0)\nabla_n u_m(\boldsymbol{q}(t)+\boldsymbol{a}t,t)\rangle$. Then $\overline{\nabla}_j u_i(t)$ is statistically equivalent to $\int_t^{t+\Delta t}\sigma_{ij}(\boldsymbol{a},t')dt'/\Delta t$, where $\sigma_{ij}(\boldsymbol{a},t)$ is a white noise:

$$\langle\sigma_{ij}(\boldsymbol{a},t)\rangle = 0, \quad \langle\sigma_{ij}(\boldsymbol{a},t)\sigma_{mn}(\boldsymbol{a},t')\rangle = \delta(t'-t)F_{ijmn}(\boldsymbol{a})\,.$$

Dropping the auxiliary time-averaging we conclude that $\lambda_1(a)$ can be found from the anisotropic Kraichnan model $\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}/\tau = \sigma\boldsymbol{R}/\tau$. For $\eta/a \ll t_\eta$ the time variation of $\nabla_j u_i(\boldsymbol{x_a}(t),t)$ is determined by the drift and one can simplify $F_{ijmn}(\boldsymbol{a}) \approx \int dt\langle\nabla_j u_i(\boldsymbol{0})\nabla_n u_m(\boldsymbol{a}t)\rangle$. The arising degeneracy $a_n F_{ijmn} = \int dt\partial_t\langle\nabla_j u_i(0)u_m(\boldsymbol{a}t)\rangle = 0$ allows to set $\sigma_{i3} \equiv 0$, where we chose $z-$axis parallel to $\boldsymbol{a}$. As a result, $\boldsymbol{r} \equiv (R_1,R_2)$ satisfies closed equation $\ddot{\boldsymbol{r}} + \dot{\boldsymbol{r}}/\tau = \hat{\sigma}\boldsymbol{r}/\tau$ where $\hat{\sigma}$ is a 2×2 matrix with $\hat{\sigma}_{ij} = \sigma_{ij}$, while $\dot{R}_3 = \sigma_{3i}r_i$. Using $\langle\nabla_j u_i(\boldsymbol{0})\nabla_n u_m(\boldsymbol{r})\rangle = \nabla_j\nabla_n S_{im}(\boldsymbol{r})/2$, where $S_{ij}(\boldsymbol{r}) = \langle[u_i(\boldsymbol{r}) - u_i(\boldsymbol{0})][u_j(\boldsymbol{r}) - u_j(\boldsymbol{0})]\rangle$, one can express F_{ijmn} with the help of second order structure function of turbulence $S_2(r) = \langle([\boldsymbol{u}(\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{0})]\cdot\boldsymbol{r}/r)^2\rangle$. After straightforward calculation one finds that $\hat{\sigma}$ obeys the statistics of $2d$ Kraichnan model with $D = D(a) = a^{-1}\int_0^\infty S_2(r)/(2r^2)\,dr$. Thus $\lambda_1(a) = D(a)^{1/3}\tilde{\lambda}_1[(D(a)\tau)^{-1/3}]/\tau^{2/3}$, for $\eta/a \ll t_\eta$ or $\sqrt{\mathrm{St}} \gg 1$. In general, $\lambda_1(a)\tau \sim (D(a)\tau)^{1/3}$ at $\mathrm{St} \gg 1$.

We now show $\lambda_1 \sim \langle\lambda_1(w)\rangle$, where the average is over the single-time statistics of w. If we let $\boldsymbol{a}$ above become slow function of time, then $p(t)$ grows with local Lyapunov exponent, $\langle\dot{p}(t)/p(t)\rangle \approx \lambda_1[a(t)]$, provided the characteristic time $\lambda_1^{-1}(a)$ of the relaxation of $\langle\dot{p}(t)/p(t)\rangle$ to its local value (see the discussion on Kraichnan model) is much smaller than the characteristic time of variations of $\boldsymbol{a}(t)$. Next, the drift velocity $\boldsymbol{w}(t)$ is determined by velocity fluctuations with $t_l \gtrsim \tau$ and thus its fixation influences weakly the statistics of velocity gradients determined by fluctuations with time-scale $t_\eta \ll \tau$. Thus, $\langle\dot{p}(t)/p(t)\rangle$ for Eq. (4) can be found first averaging over the gradients at fixed $\boldsymbol{w}(t)$ and then averaging over $\boldsymbol{w}(t)$. The first averaging gives $\lambda_1[w(t)]$ provided $\lambda_1^{-1}[w(t)]$ is much smaller than the characteristic time τ of variations of $\boldsymbol{w}(t)$. However, under the latter condition Kraichnan model gives $\lambda_1[w(t)] = \lambda^0 D^{1/3}[w(t)]/\tau^{2/3}$. Using the expression for $D(a)$ and averaging over $\boldsymbol{w}$, we find

$$\lambda_1 \approx G, \quad \text{for} \quad G \equiv \lambda^0\left\langle w^{-1/3}\right\rangle\left(\int_0^\infty\frac{S_2(r)dr}{2\tau^2 r^2}\right)^{1/3} \gg \frac{1}{\tau}. \quad (6)$$

For $\lambda_1^{-1}[w(t)] \sim \tau$ the above procedure gives the order of magnitude estimate $\lambda_1 \sim \langle\lambda_1(w)\rangle$. Using the results on Kraichnan model, this gives $\lambda_1\tau \sim 1$ at $G\tau \sim 1$. Moderate order moments $\langle w^{-1/3}\rangle$ and $\int_0^\infty S_2(r)dr/r^2$ entering G are expected to be well described by K41 at realistic Re. Indeed using for the estimates the multifractal model [2] to analyze $\int_0^\infty S_2(r)dr/r^2$ and express the anomalous exponents of w via empirical values of spatial anomalous exponents [18], one finds that intermittency becomes important at Re well above 10^{15}. Thus the above use of K41 for order of magnitude estimates is self-consistent. For G we obtain $G\tau \sim \mathrm{St}^{1/6}$. We find the behavior of λ_1 similar to Kraichnan model. The region of weak inertia, $\lambda_1 \approx \lambda_1^{turb} \ll 1/\tau$, is separated from the region of strong inertia $\lambda_1\tau \gg 1$ where Eq. (6) holds, by the long crossover $\mathrm{St} \gtrsim \mathrm{St}^{1/6} \sim 1$ where $\lambda_1\tau \sim 1$ is a slowly varying function of St (note $\lambda_1\tau \sim \langle(D(w)\tau)^{1/3}\tilde{\lambda}_1[((D(w)\tau)^{-1/3})]\rangle$ at $\sqrt{\mathrm{St}} \gg 1$). The limit $\mathrm{St}^{1/6} \gg 1$ at $\tau \ll t_L$ means very large Re and is of theoretical value mainly. Summarizing:

$$\lambda_1/\lambda_1^{turb} \sim \mathrm{St}^{-5/6} \qquad \text{for} \;\; \mathrm{St} \gtrsim 1, \qquad (7)$$

The decay of $\lambda_1/\lambda_1^{turb}$ at increasing τ is faster than in Kraichnan model because of $D \sim \tau^{-1/2}$.

We now consider the growth of $\boldsymbol{R}$ in the inertial range, at $\eta \lesssim R \ll l_\tau$. In contrast to Richardson's law for fluid particles $R(t) \sim \epsilon^{1/2}t^{3/2}$ [2, 19], K41 dimensional analysis does not fix the separation law for inertial particles,

due to the additional time-scale τ. We shall assume moderate $\dot{\boldsymbol{R}}(0)$ not to have mere ballistic motion, e. g. the analysis below applies to $R(0) \sim \eta$, $\dot{R}(0) \sim \lambda_1 \eta$, holding after separation at $R \ll \eta$. As we will see, $R(t)$ reaches l_τ within $t \sim \tau$, so to study separation at $R(t) \ll l_\tau$ we assume $t \ll \tau$. The "friction" term $\dot{\boldsymbol{R}}/\tau$ in Eq. (3) produces negligible effect over $t \ll \tau$ and can be omitted. Also $\delta \boldsymbol{u}(\boldsymbol{R}) \approx \boldsymbol{u}(\boldsymbol{q}(t) + \boldsymbol{w}t + \boldsymbol{R}(t), t) - \boldsymbol{u}(\boldsymbol{q}(t) + \boldsymbol{w}t, t)$, where $\boldsymbol{w} \equiv \boldsymbol{w}(0)$. The correlation time $t_c(R)$ of $\delta \boldsymbol{u}(R)$ is due to the drift, $t_c(R) \sim R/w \lesssim t_R$. As we verify later, the time-scale $\tau_c(R)$ of variations of R obeys $\tau_c(R) \gg t_c(R)$. Proceeding like in the viscous range, we introduce Langevin description of $\delta u_i(\boldsymbol{R})$, substituting it by white noise $D_{ij}(\boldsymbol{R})\gamma_j$, where $\langle \gamma_i(t)\gamma_j(t')\rangle = \delta_{ij}\delta(t-t')$. Here $D_{ik}(\boldsymbol{R})D_{jk}(\boldsymbol{R}) = \int dt \langle [u_i(\boldsymbol{R}) - u_i(0)][u_j(\boldsymbol{w}t + \boldsymbol{R}) - u_j(\boldsymbol{w}t)]\rangle$ to provide the correct dispersion of the time-averaged $\delta \boldsymbol{u}(\boldsymbol{R})$ [17]. We assumed for simplicity $R/w \ll t_R$ or $(l_\tau/R)^{1/3} \gg 1$ (we use K41 as at $R \ll \eta$), so that the time correlations of $\delta \boldsymbol{u}(\boldsymbol{R})$ are determined by the drift (cf. to $\eta/a \ll t_\eta$ at $R \ll \eta$). The above Kraichnan model for particles is not the same as used usually to model turbulence in the inertial range [19]: $t_c(R)$ depends on R differently than t_R. Noting $D_{ik}(\boldsymbol{R})D_{jk}(\boldsymbol{R}) \sim S_2(R)t_c(R) \sim S_2(R)R/w$, we conclude that the dependence on ϵ and τ in $\ddot{R}_i = D_{ij}(\boldsymbol{R}, \boldsymbol{w})\gamma_j/\tau$ is via single parameter $\epsilon^{2/3}/w\tau^2 \sim l_\tau^{1/3}/\tau^3$. Now dimensional analysis is enough to fix the answer. We find $\tau_c(R) \sim \tau(R/l_\tau)^{1/9}$, so the applicability condition $\tau_c(R) \gg R/w$ gives $(R/l_\tau)^{8/9} \ll 1$, close to just $R \ll l_\tau$. At $t \gg \tau_c[R(0)]$ the initial condition is forgotten (we assume explosive separation characteristic of the inertial range [19]) and $R(t)$ depends only on t and $l_\tau^{1/3}/\tau^3$ giving

$$R(t) \sim l_\tau (t/\tau)^9, \quad \tau_c[R(0)] \ll t \ll \tau. \tag{8}$$

In Kraichnan model, the power-law exponent for dispersion of fluid particles grows indefinitely as the flow becomes less rough (for smooth flow separation is exponential) [19]. Thus in Eq. (8) exponent larger than in Richardson's law can be attributed to effectively smoother turbulence felt by particles. As $R(0) \gtrsim \eta$, observability of the power-law entails $\tau \gg \tau_c[R(0)] \gtrsim \tau_c(\eta)$. This gives $(l_\tau/\eta)^{1/9} \sim \mathrm{St}^{1/6} \gg 1$, equivalent to the natural "forgetting" condition $\lambda_1^{-1} \ll \tau$. At $\mathrm{St}^{1/6} \sim 1$, the time of forgetting of the initial condition obeys $\tau_c[R(0)] \sim \tau$ so $R(t)$ at $t \ll \tau$ depends on the details of initial conditions. Eq. (8) then can be used as order of magnitude estimate at $t \sim \tau$ giving $R(\tau) \sim l_\tau$. This is expectable - for *fluid* particles the time of separation to l_τ is of order τ and determined by the stage of evolution with $R(t) \sim l_\tau$ where fluid and inertial particles behave similarly.

To summarize, we have shown that the difference of the single particle velocity and the local velocity of the flow grows with inertia as the Lagrangian velocity increment of turbulence at time τ. While at $\mathrm{St} \ll 1$ particles disperse like fluid parfticles, at $\mathrm{St} \gg 1$ there is a scale $l_\tau \sim \eta \mathrm{St}^{3/2}$ below which particle dispersion obeys laws special for inertial particles and contribution of not too intermittent events can be described by white noise. The Lyapunov exponent $\lambda_1 \sim \lambda_1^{turb}/\mathrm{St}^{5/6}$ is estimated as the average of the Lyapunov exponent for constant drift problem over the drift velocity, the estimate becoming exact at $\mathrm{St}^{1/6} \gg 1$. In the inertial range, $\eta \lesssim R \ll l_\tau$, the analogue of Richardson's law is $R(t) \sim l_\tau(t/\tau)^9$. The law, observable only at $\mathrm{St}^{1/6} \gg 1$, shows property expected at any $\mathrm{St} \gg 1$: explosive separation to l_τ within $t \sim \tau$, closer than Richardson's law to exponential separation for smooth flows. The smoothing is by the effective time-averaging of turbulent velocity difference $\delta \boldsymbol{u}(\boldsymbol{R})$ driving the separation. Our treatment can be easily generalized to incorporate constant gravitational or electric field, where constant drift problem emerges [14, 17] (e.g. for gravity acceleration $\boldsymbol{g}$ one finds $\boldsymbol{a} = \boldsymbol{g}\tau$ and $\lambda_1 = \lambda_1[g\tau]$). The main qualitative result of the work is that collective drift of inertial particles through the flow makes their relative motion subject to Langevin description.

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